NOTE ON A PAPER OF B. GRÜNBAUM ON ACYCLIC COLORINGS

BY

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ABSTRACT

The aim of this short note is to improve some recent results of B. Grünbaum by some remarks. We use Grünbaum's notations.

1.

Grünbaum gives an example of a planar graph with 14 vertices which is not (1,3)-colorable and mentions that this is the smallest known planar graph having this property. It is easy to verify that the graph G_1 in Fig. 1 below with 11 vertices is also not (1,3)-colorable. It may be shown that 11 is the minimum number of vertices (obviously one has to check only maximal planar graphs without vertices of degree 3 and there are only 20 such graphs with less than 11 vertices).

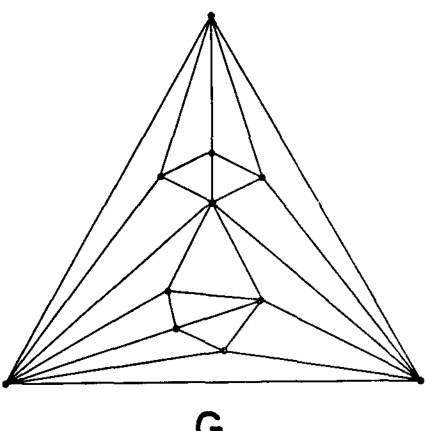

Fig. 1

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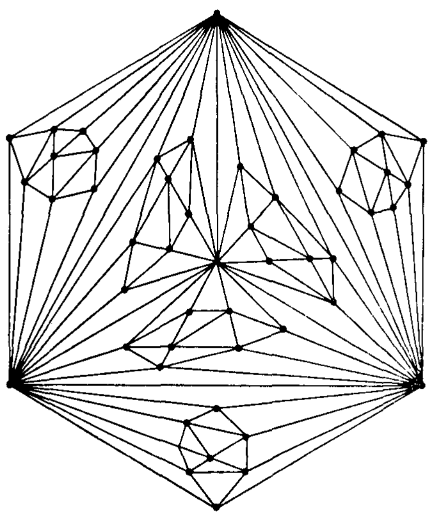
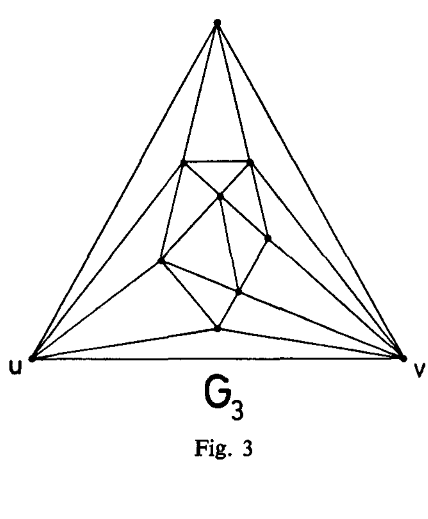
2.

The graph G_2 shown in Fig. 2 is not $(1,1,2)$-colorable, thus giving an affirmative answer to a conjecture of Grünbaum (compare remark (4) in [1]). G_2 contains as subgraphs a 4-clique and six copies of G_3 (see Fig. 3) combined in such a manner that each pair of vertices of the 4-clique is the basis pair of vertices of a copy of G_3. G_3 is a subgraph of the graph of Fig. 8 of [1] and has the property that any 4-coloring of G_3 yields a 4-circuit C 2-colored with the colors of the basis vertices u, v of G_3 (C does not necessarily contain both vertices u, v themselves). Now it is obvious that G_2 is not $(1,1,2)$-colorable.



G_2

Fig. 2



G_3

Fig. 3

3.

Finally we consider the problem mentioned in remark (12)(i) of [1]. Concerning the special case of diagonalized polygons we prove:

Each diagonalized polygon has a 6-coloring in which each bicolored path involves at most three vertices.

To prove this we note first that each diagonalized polygon is isomorphic to a subgraph of some standard polygon Q_n ($n = 0,1,2,\cdots$), where Q_0 is a triangle and Q_n is obtained from Q_{n-1} in the following manner: To each edge e of Q_{n-1}, which is adjacent to the unbounded face F of Q_{n-1}, we associate a new vertex $v(e)$ lying in F and we connect $v(e)$ with both vertices of e by edges, thus attaching $3 \cdot 2^{n-1}$ new triangles to Q_{n-1} (see Fig. 4). Now it is sufficient to give a 6-coloring of Q_n with the required property. For each k we consider Q_k as a subgraph of Q_{k+1} as indicated by the construction above. Thus we have

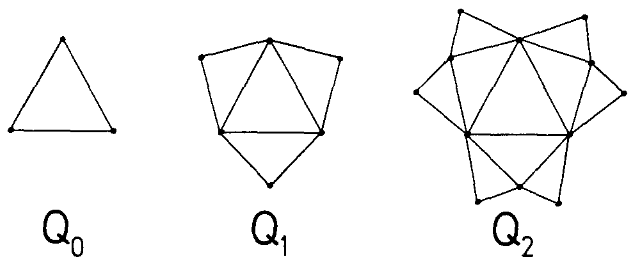


Fig. 4

$$Q_0 \subset Q_1 \subset Q_2 \subset \cdots \subset Q_k \subset Q_{k+1} \subset \cdots$$

and each vertex v of Q_n may be supplied with a rank $r(v)$ by setting $r(v) = k$ iff $v \in Q_k \setminus Q_{k-1}$ and $k > 0$ and $r(v) = 0$ iff $v \in Q_0$. Then each vertex v of Q_n has just two neighbours with rank $\leqq r(v)$, where equality holds only if $r(v) = 0$.

Now a 6-coloring of Q_n will be defined by induction. First we assign different colors to all six vertices of Q_1. Now let $r(v) = k > 1$ and assume Q_{k-1} to be 6-colored. v has only two neighbors v_1, v_2 with $r(v_i) < r(v)$ and because of $r(v) > 1$ we have $r(v_1) \neq r(v_2)$, say $r(v_1) < r(v_2)$. Again v_1 has two neighbors with rank $\leqq r(v_1)$ and likewise v_2 has two neighbors with rank $\leqq r(v_2)$, one of them being v_1 (and the second coinciding possibly with a neighbor of v_1). Thus this set of neighbors and of second order neighbors of v with decreasing rank contains at most five vertices and v shall get a color not occurring on these vertices. Since the vertices of rank k are independent the 6-coloring of Q_{k-1} may be extended to all vertices of rank k in this way. So we get a 6-coloring of Q_n and this 6-coloring has the desired property. Consider any path in Q_n of length at least 3 with vertices $v_1, v_2, \cdots, v_j$ ($j \geqq 4$). Then there is an index i either with $r(v_{i-1}) \leqq r(v_i) \geqq r(v_{i+1})$ or with $r(v_{i-1}) \leqq r(v_i) < r(v_{i+1})$ or with $r(v_{i-1}) > r(v_i) \geqq r(v_{i+1})$. In the first case, v_{i-1}, v_i, v_{i+1} are vertices of a triangle thus having three colors. In the latter cases, these vertices form a path of length 2 with monotonously increasing (or decreasing) rank; therefore, these vertices have different colors by construction of the coloring.

For diagonalized polygons, six is the best number; it is easy to see that Q_n needs six colors if $n \geqq 6$. Simple examples show that a planar graph may need more than six colors for such a coloring.

Reference

1. B. Grünbaum, *Acyclic colorings of planar graphs*, Israel J. Math. **14** (1973), 390-408.

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